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Note

Upwind Differencing, False Scaling, and Nonphysical Solutions to the Driven Cavity Problem*

We show that, for multi-dimensional viscous flow computations, the use of upwind finite difference schemes can alter the natural length scales. This false scaling is related to, but distinct from, the artificial viscosity introduced by upwind schemes. We show that this false scaling can account for certain nonphysical solutions which have been computed for the driven cavity problem.

1. Introduction

It is widely known that the use of upwind finite difference schemes for equations describing viscous flow can introduce substantial amounts of artificial viscosity at high Reynolds numbers (see, e.g., Bozeman and Dalton [1] and de Vahl Davis and Mallinson [2]). The purpose of this paper is to show that in multidimensional problems, upwind differencing can also alter the natural length scales of the problem. In particular, in Section 3 we show how this false scaling can account for certain nonphysical solutions which have been computed for the driven cavity problem.

2. False Scaling

We begin by considering a single homogeneous elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \tag{2.1}$$

on a rectangular domain

$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant \bar{y},$$

with $u(x, y)$ specified on the boundary. We assume that a and b are positive constants. The upwind difference scheme for (2.1) is

$$\begin{aligned}(u_{i+1j} - 2u_{ij} + u_{i-1j})/\Delta x^2 &+ (u_{ij+1} - 2u_{ij} + u_{ij-1})/\Delta y^2 \\ &+ a(u_{i+1j} - u_{ij})/\Delta x + b(u_{ij+1} - u_{ij})/\Delta y = 0,\end{aligned} \tag{2.2}$$

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which is the same as

$$(1 + \tfrac{1}{2}a\,\Delta x)(u_{i+1j} - 2u_{ij} + u_{i-1j})/\Delta x^2 + (1 + \tfrac{1}{2}b\,\Delta y)(u_{ij+1} - 2u_{ij} + u_{ij-1})/\Delta y^2$$
$$+ a(u_{i+1j} - u_{i-1j})/2\,\Delta x + b(u_{ij+1} - u_{ij-1})/2\,\Delta y = 0. \tag{2.3}$$

Now, for a fixed value of Δx, (2.3) can be regarded as a central difference approximation to

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} + \beta^2 \frac{\partial^2 u}{\partial y^2} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0, \tag{2.4}$$

where

$$\alpha^2 = 1 + \tfrac{1}{2}a\,\Delta x \qquad \text{and} \qquad \beta^2 = 1 + \tfrac{1}{2}b\,\Delta y.$$

If we change variables in (2.4) by $y' = y\alpha/\beta$, we obtain, after dividing by α^2,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y'^2} + \frac{a}{\alpha^2}\frac{\partial u}{\partial x} + \frac{b}{\alpha\beta}\frac{\partial u}{\partial y'} = 0. \tag{2.5}$$

If we define the Reynolds number of (2.1) as $R = \sqrt{a^2 + b^2}$, then the Reynolds number of (2.5) is $R' = \sqrt{(a/\alpha)^2 + (b/\beta)^2}/\alpha^2$, and so $R' < R$. This is the effect of artificial viscosity. Moreover, the rectangular region for (2.1) has the height, or aspect ratio, of $\bar{y}$, and that for (2.5) is $\bar{y}\alpha/\beta$. We describe this change in aspect ratio as *false scaling.*

Thus, solving (2.1) by upwind differences for given values of Δx and Δy is equivalent to solving (2.5) by central differences, where (2.5) has both a lower Reynolds number and different aspect ratio than (2.1). Since central differenting is second-order accurate and upwind differencing is only first-order accurate, we claim that the solution to (2.2) for given Δx and Δy is closer to the solution of (2.5) than it is to the solution of (2.1). This is indeed true for the equivalent one-dimensional problem for a wide range of parameters, as is shown in the Appendix.

3. False Scaling and the Driven Cavity Problem

We now look at the driven cavity problem to study the effect of the false scaling (see Bozeman and Dalton [1] for a description of the problem). The equations is convective form are

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = -\omega, \qquad \frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + R\left(\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}\right) = 0, \tag{3.1}$$

on the square $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. The Reynolds number is R and ψ and ω are the streamfunction and vorticity, respectively. The streamfunction and its normal derivative are specified on the boundary. The top wall, at $y=1$, moves with unit speed to the right. The other walls are fixed. Because of the nonlinearity of the system (3.1), it is impossible to analyze rigorously the effect of upwind differencing. Consider as a model, however,

$$\frac{\partial^2 y}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=-\omega, \qquad \alpha^2\frac{\partial^2 \omega}{\partial x^2}+\beta^2\frac{\partial^2 \omega}{\partial y^2}+R\left(\frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}\right)=0, \tag{3.2}$$

with $\alpha>\beta>1$. The constant α is taken to be greater than β since it is assumed that the large velocity in the x-direction near the top driving wall would give a larger contribution to the false diffusion than would the y-components of the velocity.

As in the previous section, let $y'=y\alpha/\beta$, and the second equation in (3.2) becomes

$$\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y'^2}+\frac{R}{\alpha\beta}\left(\frac{\partial \psi}{\partial y'}\frac{\partial \omega}{\partial x}-\frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y'}\right)=0,$$

which has an effective Reynolds number of $R/\alpha\beta$, and the domain has an effective aspect ratio of $\alpha/\beta>1$.

The solution of Eqs. (3.1) for a square driven cavity is characterized by a single large central vortex for any value of R (Pan and Acrivos [5], Bozeman and Dalton [1], de Vahl Davis and Mallinson [2], and Keller and Schreiber [8]). If the aspect ratio of the cavity is greater than about 1.6, the solution can have (at least) two large vortices (Pan and Acrivos [5] and Bozeman and Dalton [1]).

The use of upwind differencing for Eq. (3.1) can, however, give solutions which have two large vortices for a square cavity (e.g., Runchel and Wolfstein [4], Gupta and Manohar [6], Shay [7], and Bozeman and Dalton [1]). This solution for the square driven cavity is almost certainly not correct as shown by the careful studies of Bozeman and Dalton [1], Keller and Schrieber [8], and others.

In light of the above analysis, the two-vortex solution for the square cavity can be explained as the result of false scaling, which makes the effective aspect ratio greater than 1.6. Indeed, the two-vortex solutions for the square cavity resemble the solutions for cavities with aspect ratio greater than 1.6 which have been squeezed onto a square.

It should be pointed out that when upwind difference schemes are applied to the divergence form of Eq. (3.1), i.e.,

$$\frac{\partial^2 \omega}{\partial x^2}+\frac{\partial^2 \omega}{\partial y^2}+R\left(\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\omega\right)-\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\omega\right)\right)=0,$$

the solutions exhibit only one large vortex for R less than 1000 (Bozeman and Dalton [1]). Why upwind differencing of the divergence form of (3.1) should not exhibit the false scaling, but only the false diffusion, is not at all clear. It could be that the false diffusion is less, or that it is distributed more evenly between the two directions so as not to give a noticeable false scaling.

4. Conclusion

Although the analysis presented here is not completely rigorous, it does appear to be useful in explaining the origin of the particular nonphysical solutions of the driven cavity problem that have been obtained by upwind difference schemes. The analysis highlights an additional danger of using upwind differencing in computing viscous fluid flow.

Appendix

Consider the one-dimensional equivalent of (2.1),

$$\frac{d^2u}{dx^2} + a\frac{du}{dx} = 0, \tag{A.1}$$

with $0 \leqslant x \leqslant 1$, $a > 0$, and $u(0) = 1$, $u(1) = 0$. The upwind finite difference scheme for (A.1) is

$$(u_{i+1} - 2u_i + u_{i-1})/\Delta x^2 + a(u_{i+1} - u_i)/\Delta x = 0, \tag{A.2}$$

with $u_0 = 1$, $u_N = 0$, and $\Delta x = 1/N$. The scheme (A.2) is equivalent to the central difference scheme

$$(1 + \tfrac{1}{2}a\,\Delta x)((u_{i+1} - 2u_i + u_{i-1})/\Delta x^2) + a(u_{i+1} - u_{i-1})/2\,\Delta x = 0, \tag{A.3}$$

and (A.3), for fixed Δx, can be regarded as an approximation to

$$\alpha^2\frac{d^2u}{dx^2} + a\frac{du}{dx} = 0, \tag{A.4}$$

with $\alpha^2 = 1 + \frac{1}{2}a\,\Delta x$, $u(0) = 1$, and $u(1) = 0$.

We shall show that, for a wide range of value of a, the solution of (A.2) and (A.3) is closer to the solution of (A.4) than it is to the solution of (A.1). This serves to justify our assertions in Sections 2 and 3.

The solution to the difference equations (A.2) and (A.3) is

$$u_i = (1 - (1 + a\,\Delta x)^{i-N})/(1 - (1 + a\,\Delta x)^{-N}), \tag{A.5}$$

and the solutions to (A.1) and (A.4) are

$$u(x) = (1 - e^{-a(1-x)})/(1 - e^{-a}) \tag{A.6}$$

and

$$u(x) = (1 - e^{-a'(1-x)})/(1 - e^{-a'}), \tag{A.7}$$

respectively, where $a' = a/\alpha^2$.

TABLE I

	u_{19}	$u(0.95)$	Rel. err.	$u(0.95)$	Rel. err.
a	From A.5	From A.6		From A.7	
1.0	0.07642	0.07715	1%	0.07642	0.0%
5.0	0.20233	0.22270	10%	0.20163	0.3%
10.0	0.33343	0.39349	18%	0.32979	1.1%
50.0	0.71429	0.91792	28%	0.67081	6.1%
100.0	0.83333	0.99326	19%	0.76035	8.8%
200.0	0.90909	0.99995	10%	0.81112	11.0%

In Table I we show the values of (A.5)–(A.7) for $\Delta x = \frac{1}{20}$ and several values of the parameter a, at $x = 1 - \Delta x = 0.95$. Also shown are the relative errors of (A.6) and (A.7) from (A.5). This is an inverse error analysis; given the discrete solution (A.5), we wish to know which continuous solution (A.6) or (A.7) is the better continuous approximation.

Note that for $1 \leqslant a \leqslant 100$, (A.7), the solution of (A.4), is closer to (A.5) than is (A.6), the solution of (A.3). On this basis, we justify our claim of Section 2 that the solutions of (2.2) are closer to the solutions of (2.5) than they are to the solutions of (2.1).

For $a \geqslant 100$, the finite difference grid does not have any grid points in the boundary layer, and it is only due to the simplicity of this example that the solution to the finite difference scheme is close to the solution of the differential equation. For more difficult problems, such grid spacings cannot be regarded as adequate since they will not resolve any features of the boundary layer.

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